1 GALILEO

Contents

1.1 Principle of Galilean relativity

Galileo Galilei

Principles of relativity address the problem of how events that occur in one place or state of motion are observed from another. And if events occurring in one place or state of motion look different from those in another, how should one determine the laws of motion?

Galileo approached this problem via a thought experiment which imagined observations of motion made inside a ship by people who could not see outside. He showed that the people isolated inside a uniformly moving ship would be *unable to determine by measurements made inside it whether they were moving!*

> . . . have the ship proceed with any speed you like, so long as the motion is uniform and not fluctuating this way and that. You will discover not the least change in all the effects named, nor could you tell from any of them whether the ship was moving or standing still.
>
> – Galileo, *Dialogue Concerning the Two Chief World Systems* [Ga1632]

Galileo's thought experiment showed that a man who is below decks on a ship cannot tell whether the ship is docked or is moving uniformly through the water at constant velocity. He may observe water dripping from a bottle, fish swimming in a tank, butterflies flying, etc. Their behaviour will be just the same, whether the ship is moving or not.

Definition 1.1.1 (Galilean transformations) *Transformations of reference location, time, orientation or state of uniform translation at constant velocity are called* ***Galilean transformations****.*

Definition 1.1.2 (Uniform rectilinear motion) *Coordinate systems related by Galilean transformations are said to be in **uniform rectilinear motion** relative to each other.*

Galileo's thought experiment led him to the following principle.

Definition 1.1.3 (Principle of Galilean relativity) *The laws of motion are independent of reference location, time, orientation or state of uniform translation at constant velocity. Hence, these laws are invariant (i.e., they do not change their forms) under Galilean transformations.*

Remark 1.1.1 (Two tenets of Galilean relativity) Galilean relativity sets out two important tenets:

- It is impossible to determine who is actually at rest.
- Objects continue in uniform motion unless acted upon.

The second tenet is known as ***Galileo's law of inertia***. It is also the basis for ***Newton's first law of motion***. □

1.2 Galilean transformations

Definition 1.2.1 (Galilean transformations) *Galilean transformations of a coordinate frame consist of space-time translations, rotations and reflections of spatial coordinates, as well as Galilean "boosts" into uniform rectilinear motion.*

In three dimensions, the Galilean transformations depend smoothly on ten real parameters, as follows:

- *Space-time translations,*

$$g_1(\mathbf{r}, t) = (\mathbf{r} + \mathbf{r}_0, t + t_0)\,.$$

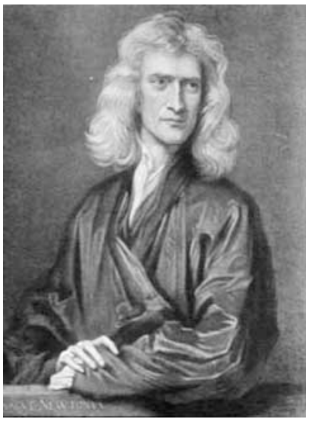
  These possess four real parameters: $(\mathbf{r}_0, t_0) \in \mathbb{R}^3 \times \mathbb{R}$, for the three dimensions of space, plus time.

2 NEWTON, LAGRANGE, HAMILTON AND THE RIGID BODY

Contents

2.1 Newton

2.1.1 Newtonian form of free rigid rotation



Isaac Newton

Definition 2.1.1 *In **free rigid rotation** a body rotates about its centre of mass and the pairwise distances between all points in the body remain fixed.*

Definition 2.1.2 *A system of coordinates fixed in a body undergoing free rigid rotation is stationary in the rotating orthonormal basis called the **body frame**, introduced by Euler [Eu1758].*

The orientation of the orthonormal frame $(\mathbf{E}_1, \mathbf{E}_2, \mathbf{E}_3)$ fixed in the rotating body relative to a basis $(\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3)$ fixed in space depends smoothly on time $t \in \mathbb{R}$. In the fixed spatial coordinate system, the body frame is seen as the moving frame

$$(O(t)\mathbf{E}_1,\, O(t)\mathbf{E}_2,\, O(t)\mathbf{E}_3)\,,$$

where $O(t) \in SO(3)$ defines the attitude of the body relative to its reference configuration according to the following matrix multiplication on its three unit vectors:

$$\mathbf{e}_a(t) = O(t)\mathbf{E}_a\,, \quad a = 1, 2, 3. \tag{2.1.1}$$

Here the unit vectors $\mathbf{e}_a(0) = \mathbf{E}_a$ with $a = 1, 2, 3$ comprise at initial time $t = 0$ an orthonormal basis of coordinates and $O(t)$ is a special ($\det O(t) = 1$) orthogonal ($O^T(t)O(t) = Id$) 3×3 matrix. That is, $O(t)$ is a continuous function defined along a curve parameterised by time t in the special orthogonal matrix group $SO(3)$. At the initial time $t = 0$, we may take $O(0) = Id$, without any loss.

Proof. The time derivative of relation (2.1.18) gives

$$\begin{aligned}\frac{d\mathbf{J}_{body}}{dt} &= \frac{d}{dt}\Big(O^{-1}(t)\mathbf{J}_{space}(t)\Big)\\ &= -O^{-1}\dot{O}\Big(O^{-1}(t)\mathbf{J}_{space}(t)\Big) + O^{-1}\underbrace{\frac{d\mathbf{J}_{space}}{dt}}_{\text{vanishes}}\\ &= -\widehat{\omega}_{body}\mathbf{J}_{body} = -\boldsymbol{\omega}_{body}\times\mathbf{J}_{body}\,,\end{aligned}\tag{2.1.20}$$

with $\widehat{\omega}_{body} := O^{-1}\dot{O} = \boldsymbol{\omega}_{body}\times$ and $\boldsymbol{\omega}_{body} = \mathbb{I}^{-1}\mathbf{J}_{body}$, which defines the ***body angular velocity***. This is the usual heuristic derivation of the dynamics for body angular momentum. ■

Remark 2.1.9 (Darwin, Coriolis and centrifugal forces) Many elementary mechanics texts make the following points about the various ***noninertial forces*** that arise in a rotating frame. For any vector $\mathbf{r}(t) = r^a(t)\mathbf{e}_a(t)$ the body and space time derivatives satisfy the first time-derivative relation, as in (2.1.17),

$$\begin{aligned}\dot{\mathbf{r}}(t) &= \dot{r}^a\mathbf{e}_a(t) + r^a\dot{\mathbf{e}}_a(t)\\ &= \dot{r}^a\mathbf{e}_a(t) + \boldsymbol{\omega}\times r^a\mathbf{e}_a(t)\\ &= \big(\dot{r}^a + \epsilon^a_{bc}\omega^b r^c\big)\mathbf{e}_a(t)\\ &=: \big(\dot{r}^a + (\boldsymbol{\omega}\times\mathbf{r})^a\big)\mathbf{e}_a(t)\,.\end{aligned}\tag{2.1.21}$$

Taking a second time derivative in this notation yields

$$\begin{aligned}\ddot{\mathbf{r}}(t) &= \big(\ddot{r}^{\,a} + (\dot{\boldsymbol{\omega}}\times\mathbf{r})^a + (\boldsymbol{\omega}\times\dot{\mathbf{r}})^a\big)\mathbf{e}_a(t) + \big(\dot{r}^a + (\boldsymbol{\omega}\times\mathbf{r})^a\big)\dot{\mathbf{e}}_a(t)\\ &= \big(\ddot{r}^{\,a} + (\dot{\boldsymbol{\omega}}\times\mathbf{r})^a + (\boldsymbol{\omega}\times\dot{\mathbf{r}})^a\big)\mathbf{e}_a(t) + \big(\boldsymbol{\omega}\times\big(\dot{\mathbf{r}} + \boldsymbol{\omega}\times\mathbf{r}\big)\big)^a\mathbf{e}_a(t)\\ &= \big(\ddot{r}^{\,a} + (\dot{\boldsymbol{\omega}}\times\mathbf{r})^a + 2(\boldsymbol{\omega}\times\dot{\mathbf{r}})^a + (\boldsymbol{\omega}\times\boldsymbol{\omega}\times\mathbf{r})\,\big)^a\mathbf{e}_a(t)\,.\end{aligned}$$

Newton's second law for the evolution of the position vector $\mathbf{r}(t)$ of a particle of mass m in a frame rotating with time-dependent angular velocity $\boldsymbol{\omega}(t)$ becomes

$$\mathbf{F}(\mathbf{r}) = m\big(\,\ddot{\mathbf{r}} + \underbrace{\dot{\boldsymbol{\omega}}\times\mathbf{r}}_{\text{Darwin}} + \underbrace{2(\boldsymbol{\omega}\times\dot{\mathbf{r}})}_{\text{Coriolis}} + \underbrace{\boldsymbol{\omega}\times(\boldsymbol{\omega}\times\mathbf{r})}_{\text{centrifugal}}\,\big)\,.\tag{2.1.22}$$

- The Darwin force is usually small; so it is often neglected.
- Only the Coriolis force depends on the velocity in the moving frame. The Coriolis force is very important in large-scale motions on Earth. For example, pressure balance with the Coriolis force dominates the (geostrophic) motion of weather systems that comprise the climate.
- The centrifugal force is important, for example, in obtaining orbital equilibria in gravitationally attracting systems.

□

Remark 2.1.10 The space and body angular velocities differ by

$$\widehat{\omega}_{body} := O^{-1}\dot{O} \quad \text{versus} \quad \widehat{\omega}_{space} := \dot{O}O^{-1} = O\widehat{\omega}_{body}O^{-1}\,.$$

Namely, $\widehat{\omega}_{body}$ is left-invariant under $O \to RO$ and $\widehat{\omega}_{space}$ is right-invariant under $O \to OR$, for any choice of matrix $R \in SO(3)$. This means that neither angular velocity depends on the initial orientation. □

Remark 2.1.11 The angular velocities $\widehat{\omega}_{body} = O^{-1}\dot{O}$ and $\widehat{\omega}_{space} = \dot{O}O^{-1}$ are respectively the left and right translations to the identity of the tangent matrix $\dot{O}(t)$ at $O(t)$. These are called the left and right tangent spaces of $SO(3)$ at its identity. □

Remark 2.1.12 Equations (2.1.19) for free rigid rotations of particle systems are prototypes of Euler's equations for the motion of a rigid body. □

2.1.2 Newtonian form of rigid-body motion

In describing rotations of a rigid body, for example a solid object occupying a spatial domain $\mathcal{B} \subset \mathbb{R}^3$, one replaces the mass-weighted sums over points in space in the previous definitions of dynamical

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2.3 Noether's theorem

2.3.1 Lie symmetries and conservation laws

Emmy Noether

Recall from Definition 1.2.3 that a Lie group depends smoothly on its parameters. (See Appendix B for more details.)

Definition 2.3.1 (Lie symmetry) *A smooth transformation of variables* $\{t, q\}$ *depending on a single parameter* s *defined by*

$$\{t, q\} \mapsto \{\, \bar{t}(t, q, s),\, \overline{q}(t, q, s)\}\,,$$

that leaves the action $S = \int L\,dt$ *invariant is called a* ***Lie symmetry*** *of the action.*

Theorem 2.3.1 (Noether's theorem) *Each Lie symmetry of the action for a Lagrangian system defined on a manifold* M *with Lagrangian* L *corresponds to a constant of the motion [No1918].*

Example 2.3.1 *Suppose the variation of the action in (2.2.3) vanishes* ($\delta S = 0$) *because of a Lie symmetry which does not preserve the endpoints. Then on solutions of the Euler–Lagrange equations, the endpoint term must vanish for another reason. For example, if the Lie symmetry leaves time invariant, so that*

$$\{t, q\} \mapsto \{\, t,\, \overline{q}(t, q, s)\}\,,$$

then the endpoint term must vanish,

$$\left[\frac{\partial L}{\partial \dot{q}}\delta q\right]_{t_1}^{t_2} = 0\,.$$

Hence, the quantity

$$A(q, \dot{q}, \delta q) = \frac{\partial L}{\partial \dot{q}^a}\,\delta q^a$$

*is a **constant of motion** for solutions of the Euler–Lagrange equations. In particular, if $\delta q^a = c^a$ for constants c^a, $a = 1, \dots, n$, that is, for spatial translations in n dimensions, then the quantities $\partial L/\partial \dot{q}^a$ (the corresponding momentum components) are constants of motion.*

Remark 2.3.1 This result first appeared in Noether [No1918]. See, e.g., [Ol2000, SaCa1981] for good discussions of the history, framework and applications of Noether's theorem. We shall see in a moment that Lie symmetries that reparameterise time may also yield constants of motion. □

2.3.2 Infinitesimal transformations of a Lie group

Definition 2.3.2 (Infinitesimal Lie transformations)

Sophus Lie

Consider the Lie group of transformations

$$\{t, q\} \mapsto \{\, \bar{t}(t,q,s),\, \bar{q}(t,q,s)\}\,,$$

and suppose the identity transformation is arranged to occur for $s = 0$. The derivatives with respect to the group parameters s at the identity,

$$\begin{aligned} \tau(t,q) &= \frac{d}{ds}\bigg|_{s=0} \bar{t}(t,q,s)\,, \\ \xi^a(t,q) &= \frac{d}{ds}\bigg|_{s=0} \bar{q}^a(t,q,s)\,, \end{aligned}$$

*are called the **infinitesimal transformations** of the action of a Lie group on the time and space variables.*

Thus, at linear order in a Taylor expansion in the group parameter s one has

$$\bar{t} = t + s\tau(t,q)\,, \quad \bar{q}^a = q^a + s\xi^a(t,q)\,, \tag{2.3.1}$$

where τ and ξ^a are functions of coordinates and time, but do not depend on velocities. Then, to first order in s the velocities of the

Thus, Euler's equations for the rigid body in $T\mathbb{R}^3$,

$$\mathbb{I}\dot{\mathbf{\Omega}} = \mathbb{I}\mathbf{\Omega} \times \mathbf{\Omega}\,, \tag{2.4.8}$$

do follow from the variational principle (2.4.3) with variations of the form (2.4.5) derived from the definition of body angular velocity $\widehat{\Omega}$.

■

Remark 2.4.1 The body angular velocity is expressed in terms of the spatial angular velocity by $\mathbf{\Omega}(t) = O^{-1}(t)\boldsymbol{\omega}(t)$. Consequently, the kinetic energy Lagrangian in (2.4.4) transforms as

$$l(\mathbf{\Omega}) = \frac{1}{2}\,\mathbf{\Omega}\cdot\mathbb{I}\mathbf{\Omega} = \frac{1}{2}\,\boldsymbol{\omega}\cdot\mathbb{I}_{space}(t)\boldsymbol{\omega} =: l_{space}(\boldsymbol{\omega})\,,$$

where

$$\mathbb{I}_{space}(t) = O(t)\mathbb{I}O^{-1}(t)\,,$$

as in (2.1.31). □

Exercise. Show that Hamilton's principle for the action

$$S(\boldsymbol{\omega}) = \int_a^b l_{space}(\boldsymbol{\omega})\,dt$$

yields conservation of spatial angular momentum

$$\boldsymbol{\pi} = \mathbb{I}_{space}(t)\boldsymbol{\omega}(t)\,.$$

Hint: First derive $\delta\mathbb{I}_{space} = [\xi, \mathbb{I}_{space}]$ with right-invariant $\xi = \delta O O^{-1}$. ★

Exercise. (Noether's theorem for the rigid body) What conservation law does Theorem 2.3.1 (Noether's theorem) imply for the rigid-body Equations (2.4.2)?

Hint: Transform the endpoint terms arising on integrating the variation δS by parts in the proof of Theorem 2.4.1 into the spatial representation by setting $\mathbf{\Xi} = O^{-1}(t)\mathbf{\Gamma}$ and $\mathbf{\Omega} = O^{-1}(t)\boldsymbol{\omega}$. ★

Remark 2.4.2 (Reconstruction of $O(t) \in SO(3)$) The Euler solution is expressed in terms of the time-dependent angular velocity vector in the body, $\mathbf{\Omega}$. The body angular velocity vector $\mathbf{\Omega}(t)$ yields the tangent vector $\dot{O}(t) \in T_{O(t)}SO(3)$ along the integral curve in the rotation group $O(t) \in SO(3)$ by the relation

$$\dot{O}(t) = O(t)\widehat{\Omega}(t)\,, \tag{2.4.9}$$

where the left-invariant skew-symmetric 3×3 matrix $\widehat{\Omega}$ is defined by the hat map (2.1.27)

$$(O^{-1}\dot{O})_{jk} = \widehat{\Omega}_{jk} = -\,\Omega_i\epsilon_{ijk}\,. \tag{2.4.10}$$

Equation (2.4.9) is the ***reconstruction formula*** for $O(t) \in SO(3)$.

Once the time dependence of $\mathbf{\Omega}(t)$ and hence $\widehat{\Omega}(t)$ is determined from the Euler equations, solving formula (2.4.9) as a linear differential equation with time-dependent coefficients yields the integral curve $O(t) \in SO(3)$ for the orientation of the rigid body. □

2.4.1 Hamilton–Pontryagin constrained variations

Formula (2.4.6) for the variation $\widehat{\Omega}$ of the skew-symmetric matrix

$$\widehat{\Omega} = O^{-1}\dot{O}$$

Proof. The proof here is the same as the proof of Manakov's commutator formula via Hamilton's principle, modulo replacing $O^{-1}\dot{O} \in so(n)$ with $g^{-1}\dot{g} \in \mathfrak{g}$. ■

Remark 2.4.7 Poincaré's observation leading to the matrix Euler–Poincaré Equation (2.4.27) was reported in two pages with no references [Po1901]. The proof above shows that the matrix Euler–Poincaré equations possess a natural variational principle. Note that if $\Omega = g^{-1}\dot{g} \in \mathfrak{g}$, then $M = \delta l/\delta\Omega \in \mathfrak{g}^*$, where the dual is defined in terms of the matrix trace pairing. □

Exercise. Retrace the proof of the variational principle for the Euler–Poincaré equation, replacing the left-invariant quantity $g^{-1}\dot{g}$ with the right-invariant quantity $\dot{g}g^{-1}$. ★

2.4.4 An isospectral eigenvalue problem for the $SO(n)$ rigid body

The solution of the $SO(n)$ rigid-body dynamics

$$\frac{dM}{dt} = [\, M \,,\, \Omega \,] \quad \text{with} \quad M = \mathbb{A}\Omega + \Omega\mathbb{A}\,,$$

for the evolution of the $n \times n$ skew-symmetric matrices M, Ω, with constant symmetric $\mathbb{A}$, is given by a similarity transformation (later to be identified as coadjoint motion),

$$M(t) = O(t)^{-1}M(0)O(t) =: \mathrm{Ad}^*_{O(t)}M(0)\,,$$

with $O(t) \in SO(n)$ and $\Omega := O^{-1}\dot{O}(t)$. Consequently, the evolution of $M(t)$ is ***isospectral***. This means that

- The initial eigenvalues of the matrix $M(0)$ are preserved by the motion; that is, $d\lambda/dt = 0$ in

  $$M(t)\psi(t) = \lambda\psi(t)\,,$$

  provided its eigenvectors $\psi \in \mathbb{R}^n$ evolve according to

  $$\psi(t) = O(t)^{-1}\psi(0)\,.$$

  The proof of this statement follows from the corresponding property of similarity transformations.

- Its matrix invariants are preserved:

  $$\frac{d}{dt}\mathrm{tr}(M - \lambda\mathrm{Id})^K = 0\,,$$

  for every non-negative integer power K.

  This is clear because the invariants of the matrix M may be expressed in terms of its eigenvalues; but these are invariant under a similarity transformation.

Proposition 2.4.5 *Isospectrality allows the quadratic rigid-body dynamics (2.4.26) on $SO(n)$ to be rephrased as a system of two coupled linear equations: the eigenvalue problem for M and an evolution equation for its eigenvectors ψ, as follows:*

$$M\psi = \lambda\psi \quad \textit{and} \quad \dot{\psi} = -\,\Omega\psi\,, \quad \textit{with} \quad \Omega = O^{-1}\dot{O}(t)\,.$$

Proof. Applying isospectrality in the time derivative of the first equation yields

$$(\,\dot{M} + [\,\Omega, M\,]\,)\psi + (M - \lambda\mathrm{Id})(\dot{\psi} + \Omega\psi) = 0\,.$$

Now substitute the second equation to recover (2.4.26). ∎