Street-Fighting Mathematics

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The Art of Educated Guessing and Opportunistic Problem Solving

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Foreword by Carver A. Mead

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For Juliet

Brief contents

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Foreword

Most of us took mathematics courses from mathematicians—Bad Idea!

Mathematicians see mathematics as an area of study in its own right. The rest of us use mathematics as a precise language for expressing relationships among quantities in the real world, and as a tool for deriving quantitative conclusions from these relationships. For that purpose, mathematics courses, as they are taught today, are seldom helpful and are often downright destructive.

As a student, I promised myself that if I ever became a teacher, I would never put a student through that kind of teaching. I have spent my life trying to find direct and transparent ways of seeing reality and trying to express these insights quantitatively, and I have never knowingly broken my promise.

With rare exceptions, the mathematics that I have found most useful was learned in science and engineering classes, on my own, or from this book. *Street-Fighting Mathematics* is a breath of fresh air. Sanjoy Mahajan teaches us, in the most friendly way, tools that work in the real world. Just when we think that a topic is obvious, he brings us up to another level. My personal favorite is the approach to the Navier–Stokes equations: so nasty that I would never even attempt a solution. But he leads us through one, gleaning gems of insight along the way.

In this little book are insights for every one of us. I have personally adopted several of the techniques that you will find here. I recommend it highly to every one of you.

—*Carver Mead*

Preface

Too much mathematical rigor teaches *rigor mortis:* the fear of making an unjustified leap even when it lands on a correct result. Instead of paralysis, have courage—shoot first and ask questions later. Although unwise as public policy, it is a valuable problem-solving philosophy, and it is the theme of this book: how to guess answers without a proof or an exact calculation.

Educated guessing and opportunistic problem solving require a toolbox. A tool, to paraphrase George Polya, is a trick I use twice. This book builds, sharpens, and demonstrates tools useful across diverse fields of human knowledge. The diverse examples help separate the tool—the general principle—from the particular applications so that you can grasp and transfer the tool to problems of particular interest to you.

The examples used to teach the tools include guessing integrals without integrating, refuting a common argument in the media, extracting physical properties from nonlinear differential equations, estimating drag forces without solving the Navier–Stokes equations, finding the shortest path that bisects a triangle, guessing bond angles, and summing infinite series whose every term is unknown and transcendental.

This book complements works such as *How to Solve It* [Polya:1957], *Mathematics and Plausible Reasoning* [Polya:1954-1,Polya:1954-2], and *The Art and Craft of Problem Solving* [Zeitz:2007]. They teach how to solve exactly stated problems exactly, whereas life often hands us partly defined problems needing only moderately accurate solutions. A calculation accurate only to a factor of 2 may show that a proposed bridge would never be built or a circuit could never work. The effort saved by not doing the precise analysis can be spent inventing promising new designs.

This book grew out of a short course of the same name that I taught for several years at MIT. The students varied widely in experience: from first-year undergraduates to graduate students ready for careers in research and teaching. The students also varied widely in specialization: from physics, mathematics, and management to electrical engineering, computer science, and biology. Despite or because of the diversity, the students seemed to benefit from the set of tools and to enjoy the diversity of illustrations and applications. I wish the same for you.

1
Dimensions

Our first street-fighting tool is dimensional analysis or, when abbreviated, dimensions. To show its diversity of application, the tool is introduced with an economics example and sharpened on examples from Newtonian mechanics and integral calculus.

1.1 Economics: The power of multinational corporations

Critics of globalization often make the following comparison [25] to prove the excessive power of multinational corporations:

> In Nigeria, a relatively economically strong country, the GDP [gross domestic product] is \$99 billion. The net worth of Exxon is \$119 billion. "When multinationals have a net worth higher than the GDP of the country in which they operate, what kind of power relationship are we talking about?" asks Laura Morosini.

Before continuing, explore the following question:

▶ *What is the most egregious fault in the comparison between Exxon and Nigeria?*

The field is competitive, but one fault stands out. It becomes evident after unpacking the meaning of GDP. A GDP of \$99 billion is shorthand for a monetary flow of \$99 billion per year. A year, which is the time for the earth to travel around the sun, is an astronomical phenomenon that has been arbitrarily chosen for measuring a social phenomenon—namely, monetary flow.

Suppose instead that economists had chosen the decade as the unit of time for measuring GDP. Then Nigeria's GDP (assuming the flow remains steady from year to year) would be roughly $1 trillion per decade and be reported as $1 trillion. Now Nigeria towers over Exxon, whose puny assets are a mere one-tenth of Nigeria's GDP. To deduce the opposite conclusion, suppose the week were the unit of time for measuring GDP. Nigeria's GDP becomes $2 billion per week, reported as $2 billion. Now puny Nigeria stands helpless before the mighty Exxon, 50-fold larger than Nigeria.

A valid economic argument cannot reach a conclusion that depends on the astronomical phenomenon chosen to measure time. The mistake lies in comparing incomparable quantities. Net worth is an amount: It has dimensions of money and is typically measured in units of dollars. GDP, however, is a flow or rate: It has dimensions of money per time and typical units of dollars per year. (A dimension is general and independent of the system of measurement, whereas the unit is how that dimension is measured in a particular system.) Comparing net worth to GDP compares a monetary amount to a monetary flow. Because their dimensions differ, the comparison is a category mistake [39] and is therefore guaranteed to generate nonsense.

Problem 1.1 Units or dimensions?

Are meters, kilograms, and seconds units or dimensions? What about energy, charge, power, and force?

A similarly flawed comparison is length per time (speed) versus length: "I walk 1.5m s^{-1}—much smaller than the Empire State building in New York, which is 300 miles high." It is nonsense. To produce the opposite but still nonsense conclusion, measure time in hours: "I walk 5400 m/hr—much larger than the Empire State building, which is 300 miles high."

I often see comparisons of corporate and national power similar to our Nigeria–Exxon example. I once wrote to one author explaining that I sympathized with his conclusion but that his argument contained a fatal dimensional mistake. He replied that I had made an interesting point but that the numerical comparison showing the country's weakness was stronger as he had written it, so he was leaving it unchanged!

A dimensionally valid comparison would compare like with like: either Nigeria's GDP with Exxon's revenues, or Exxon's net worth with Nigeria's net worth. Because net worths of countries are not often tabulated, whereas corporate revenues are widely available, try comparing Exxon's annual revenues with Nigeria's GDP. By 2006, Exxon had become Exxon Mobil with annual revenues of roughly $350 billion—almost twice Nigeria's 2006 GDP of $200 billion. This valid comparison is stronger than the flawed one, so retaining the flawed comparison was not even expedient!

That compared quantities must have identical dimensions is a necessary condition for making valid comparisons, but it is not sufficient. A costly illustration is the 1999 Mars Climate Orbiter (MCO), which crashed into the surface of Mars rather than slipping into orbit around it. The cause, according to the Mishap Investigation Board (MIB), was a mismatch between English and metric units [MarsClimateOrbiter:1999]:

> The MCO MIB has determined that the root cause for the loss of the MCO spacecraft was the failure to use metric units in the coding of a ground software file, Small Forces, used in trajectory models. Specifically, thruster performance data in English units instead of metric units was used in the software application code titled SM_FORCES (small forces). A file called Angular Momentum Desaturation (AMD) contained the output data from the SM_FORCES software. The data in the AMD file was required to be in metric units per existing software interface documentation, and the trajectory modelers assumed the data was provided in metric units per the requirements.

Make sure to mind your dimensions and units.

Problem 1.2 Finding bad comparisons

Look for everyday comparisons—for example, on the news, in the newspaper, or on the Internet—that are dimensionally faulty.

1.2 Newtonian mechanics: Free fall

Dimensions are useful not just to debunk incorrect arguments but also to generate correct ones. To do so, the quantities in a problem need to have dimensions. As a contrary example showing what not to do, here is how many calculus textbooks introduce a classic problem in motion:

> A ball initially at rest falls from a height of h **feet** and hits the ground at a speed of v **feet per second**. Find v assuming a gravitational acceleration of g **feet per second squared** and neglecting air resistance.

The units such as feet or feet per second are highlighted in boldface because their inclusion is so frequent as to otherwise escape notice, and their inclusion creates a significant problem. Because the height is h feet, the variable h does not contain the units of height: h is therefore dimensionless. (For h to have dimensions, the problem would instead state simply that the ball falls from a height h; then the dimension of length would belong to h.) A similar explicit specification of units means that the variables g and v are also dimensionless. Because g, h, and v are dimensionless, any comparison of v with quantities derived from g and h is a comparison between dimensionless quantities. It is therefore always dimensionally valid, so dimensional analysis cannot help us guess the impact speed.

Giving up the valuable tool of dimensions is like fighting with one hand tied behind our back. Thereby constrained, we must instead solve the following differential equation with initial conditions:

$$\frac{d^2y}{dt^2} = -g, \text{ with } y(0) = h \text{ and } dy/dt = 0 \text{ at } t = 0, \tag{1.1}$$

where $y(t)$ is the ball's height, dy/dt is the ball's velocity, and g is the gravitational acceleration.

Problem 1.3 Calculus solution

Use calculus to show that the free-fall differential equation $d^2y/dt^2 = -g$ with initial conditions $y(0) = h$ and $dy/dt = 0$ at $t = 0$ has the following solution:

$$\frac{dy}{dt} = -gt \quad \text{and} \quad y = -\frac{1}{2}gt^2 + h. \tag{1.2}$$

Using the solutions for the ball's position and velocity in, what is the impact speed?

When $y(t) = 0$, the ball meets the ground. Thus the impact time t_0 is $\sqrt{2h/g}$. The impact velocity is $-gt_0$ or $-\sqrt{2gh}$. Therefore the impact speed (the unsigned velocity) is $\sqrt{2gh}$.

This analysis invites several algebra mistakes: forgetting to take a square root when solving for t_0, or dividing rather than multiplying by g when finding the impact velocity. Practice—in other words, making and correcting many mistakes—reduces their prevalence in simple problems, but complex problems with many steps remain minefields. We would like less error-prone methods.

One robust alternative is the method of dimensional analysis. But this tool requires that at least one quantity among v, g, and h have dimensions. Otherwise, every candidate impact speed, no matter how absurd, equates dimensionless quantities and therefore has valid dimensions.

Therefore, let's restate the free-fall problem so that the quantities retain their dimensions:

> A ball initially at rest falls from a height h and hits the ground at speed v. Find v assuming a gravitational acceleration g and neglecting air resistance.

The restatement is, first, shorter and crisper than the original phrasing:

> A ball initially at rest falls from a height of h feet and hits the ground at a speed of v feet per second. Find v assuming a gravitational acceleration of g feet per second squared and neglecting air resistance.

Second, the restatement is more general. It makes no assumption about the system of units, so it is useful even if meters, cubits, or furlongs are the unit of length. Most importantly, the restatement gives dimensions to h, g, and v. Their dimensions will almost uniquely determine the impact speed—without our needing to solve a differential equation.

The dimensions of height h are simply length or, for short, L. The dimensions of gravitational acceleration g are length per time squared or LT^{-2}, where T represents the dimension of time. A speed has dimensions of LT^{-1}, so v is a function of g and h with dimensions of LT^{-1}.

Problem 1.4 Dimensions of familiar quantities

In terms of the basic dimensions length L, mass M, and time T, what are the dimensions of energy, power, and torque?

▶ *What combination of g and h has dimensions of speed?*

The combination $\sqrt{gh}$ has dimensions of speed.

$$(\underbrace{LT^{-2}}_{g} \times \underbrace{L}_{h})^{1/2} = \sqrt{L^2T^{-2}} = \underbrace{LT^{-1}}_{\text{speed}}. \tag{1.3}$$

▶ *Is $\sqrt{gh}$ the only combination of g and h with dimensions of speed?*

In order to decide whether $\sqrt{gh}$ is the only possibility, use constraint propagation [Stallman:1976]. The strongest constraint is that the combination of g and h, being a speed, should have dimensions of inverse time (T^{-1}). Because h contains no dimensions of time, it cannot help construct T^{-1}. Because g contains T^{-2}, the T^{-1} must come from $\sqrt{g}$. The second constraint is that the combination contain L^1. The $\sqrt{g}$ already contributes $L^{1/2}$, so the missing $L^{1/2}$ must come from $\sqrt{h}$. The two constraints thereby determine uniquely how g and h appear in the impact speed v.

The exact expression for v is, however, not unique. It could be $\sqrt{gh}$, $\sqrt{2gh}$, or, in general, $\sqrt{gh} \times$ dimensionless constant. The idiom of multiplication by a dimensionless constant occurs frequently and deserves a compact notation akin to the equals sign:

$$v \sim \sqrt{gh}. \tag{1.4}$$

Including this $\sim$ notation, we have several species of equality:

$\propto$	equality except perhaps for a factor with dimensions,	
$\sim$	equality except perhaps for a factor without dimensions,	(1.5)
$\approx$	equality except perhaps for a factor close to 1.	

The exact impact speed is $\sqrt{2gh}$, so the dimensions result $\sqrt{gh}$ contains the entire functional dependence! It lacks only the dimensionless factor $\sqrt{2}$, and these factors are often unimportant. In this example, the height might vary from a few centimeters (a flea hopping) to a few meters (a cat jumping from a ledge). The factor-of-100 variation in height contributes a factor-of-10 variation in impact speed. Similarly, the gravitational acceleration might vary from $0.27\,\mathrm{mpss}$ (on the asteroid Ceres) to $25\,\mathrm{mpss}$ (on Jupiter). The factor-of-100 variation in g contributes another factor-of-10 variation in impact speed. Much variation in the impact speed, therefore, comes not from the dimensionless factor $\sqrt{2}$ but rather from the symbolic factors—which are computed exactly by dimensional analysis.

Furthermore, not calculating the exact answer can be an advantage. Exact answers have all factors and terms, permitting less important information, such as the dimensionless factor $\sqrt{2}$, to obscure important information such as $\sqrt{gh}$. As William James advised, "The art of being wise is the art of knowing what to overlook"

Problem 1.5 Vertical throw

You throw a ball directly upward with speed v_0. Use dimensional analysis to estimate how long the ball takes to return to your hand (neglecting air resistance). Then find the exact time by solving the free-fall differential equation. What dimensionless factor was missing from the dimensional-analysis result?

1.3 Guessing integrals

The analysis of free fall (1.2) shows the value of not separating dimensioned quantities from their units. However, what if the quantities are dimensionless, such as the 5 and x in the following Gaussian integral:

$$\int_{-\infty}^{\infty} e^{-5x^2}\, dx\,? \tag{1.6}$$

Alternatively, the dimensions might be unspecified—a common case in mathematics because it is a universal language. For example, probability theory uses the Gaussian integral

$$\int_{x_1}^{x_2} e^{-x^2/2\sigma^2}\, dx, \tag{1.7}$$

where x could be height, detector error, or much else. Thermal physics uses the similar integral

$$\int e^{-\frac{1}{2}mv^2/kT}\, dv, \tag{1.8}$$

where v is a molecular speed. Mathematics, as the common language, studies their common form $\int e^{-\alpha x^2}$ without specifying the dimensions of α and x. The lack of specificity gives mathematics its power of abstraction, but it makes using dimensional analysis difficult.

▶ *How can dimensional analysis be applied without losing the benefits of mathematical abstraction?*

1.4 Summary and further problems

Do not add apples to oranges: Every term in an equation or sum must have identical dimensions! This restriction is a powerful tool. It helps us to evaluate integrals without integrating and to predict the solutions of differential equations. Here are further problems to practice this tool.

Problem 1.6 Integrals using dimensions

Use dimensional analysis to find $\int_0^{\infty} e^{-ax}\, dx$ and $\int \frac{dx}{x^2+a^2}$. A useful result is

$$\int \frac{dx}{x^2+1} = \arctan x + C. \tag{1.9}$$

Problem 1.7 Stefan–Boltzmann law

Blackbody radiation is an electromagnetic phenomenon, so the radiation intensity depends on the speed of light c. It is also a thermal phenomenon, so it depends on the thermal energy $k_B T$, where T is the object's temperature and k_B is Boltzmann's constant. And it is a quantum phenomenon, so it depends on Planck's constant $\hbar$. Thus the blackbody-radiation intensity I depends on c, $k_B T$, and $\hbar$. Use dimensional analysis to show that $I \propto T^4$ and to find the constant of proportionality σ. Then look up the missing dimensionless constant. (These results are used in 1.1.)

Problem 1.8 Arcsine integral

Use dimensional analysis to find $\int \sqrt{1-3x^2}\,dx$. A useful result is

$$\int \sqrt{1-x^2}\,dx = \frac{\arcsin x}{2} + \frac{x\sqrt{1-x^2}}{2} + C, \tag{1.10}$$

Problem 1.9 Related rates

Water is poured into a large inverted cone (with a 90° opening angle) at a rate $dV/dt = 10\text{miles}^3\,\text{s}^{-1}$. When the water depth is $h = 5\text{miles}$, estimate the rate at which the depth is increasing. Then use calculus to find the exact rate.

2
Easy cases

A correct solution works in all cases, including the easy ones. This maxim underlies the second tool—the method of easy cases. It will help us guess integrals, deduce volumes, and solve exacting differential equations.

2.1 Gaussian integral revisited

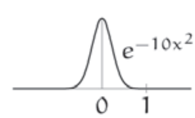
As the first application, let's revisit the Gaussian integral from Section 1.2,

$$\int_{-\infty}^{\infty} e^{-\alpha x^2}\,dx. \tag{2.1}$$

▶ *Is the integral $\sqrt{\pi\alpha}$ or $\sqrt{\pi/\alpha}$?*

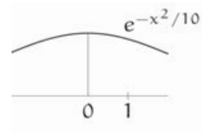
The correct choice must work for all $\alpha \geq 0$. At this range's endpoints ($\alpha = \infty$ and $\alpha = 0$), the integral is easy to evaluate.

▶ *What is the integral when $\alpha = \infty$?*



As the first easy case, increase α to ∞. Then $-\alpha x^2$ becomes very negative, even when x is tiny. The exponential of a large negative number is tiny, so the bell curve narrows to a sliver, and its area shrinks to zero. Therefore, as $\alpha \to \infty$ the integral shrinks to zero. This result refutes the option $\sqrt{\pi\alpha}$, which is infinite when $\alpha = \infty$; and it supports the option $\sqrt{\pi/\alpha}$, which is zero when $\alpha = \infty$.

▶ *What is the integral when $\alpha = 0$?*



In the $\alpha = 0$ extreme, the bell curve flattens into a horizontal line with unit height. Its area, integrated over the infinite range, is infinite. This result refutes the $\sqrt{\pi\alpha}$ option, which is zero when $\alpha = 0$; and it supports the $\sqrt{\pi/\alpha}$ option, which is infinity when $\alpha = 0$. Thus the $\sqrt{\pi\alpha}$ option fails both easy-cases tests, and the $\sqrt{\pi/\alpha}$ option passes both easy-cases tests.

If these two options were the only options, we would choose $\sqrt{\pi/\alpha}$. However, if a third option were $\sqrt{2/\alpha}$, how could you decide between it and $\sqrt{\pi/\alpha}$? Both options pass both easy-cases tests; they also have identical dimensions. The choice looks difficult.

To choose, try a third easy case: $\alpha = 1$. Then the integral simplifies to

$$\int_{-\infty}^{\infty} e^{-x^2}\, dx. \tag{2.2}$$

This classic integral can be evaluated in closed form by using polar coordinates, but that method also requires a trick with few other applications (textbooks on multivariable calculus give the gory details). A less elegant but more general approach is to evaluate the integral numerically and to use the approximate value to guess the closed form.

Therefore, replace the smooth curve e^{-x^2} with a curve having n line segments. This piecewise-linear approximation turns the area into a sum of n trapezoids. As n approaches infinity, the area of the trapezoids more and more closely approaches the area under the smooth curve.

n	Area
10	2.07326300569564
20	1.77263720482665
30	1.77245385170978
40	1.77245385090552
50	1.77245385090552

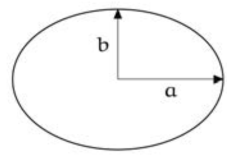
The table gives the area under the curve in the range $x = -10 \ldots 10$, after dividing the curve into n line segments. The areas settle onto a stable value, and it looks familiar. It begins with 1.7, which might arise from $\sqrt{3}$. However, it continues as 1.77, which is too large to be $\sqrt{3}$. Fortunately, π is slightly larger than 3, so the area might be converging to $\sqrt{\pi}$.

Let's check by comparing the squared area against π:

$$\begin{aligned} 1.77245385090552^2 &\approx 3.14159265358980, \\ \pi &\approx 3.14159265358979. \end{aligned} \tag{2.3}$$

2.2 Plane geometry: The area of an ellipse

The second application of easy cases is from plane geometry: the area of an ellipse. This ellipse has semimajor axis a and semiminor axis b. For its area A consider the following candidates:



(a) ab^2 (b) $a^2 + b^2$ (c) a^3/b (d) $2ab$ (e) πab.

▶ *What are the merits or drawbacks of each candidate?*

The candidate $A = ab^2$ has dimensions of L^3, whereas an area must have dimensions of L^2. Thus ab^2 must be wrong.

Problem 2.1 Area by calculus

Use integration to show that $A = \pi ab$.

Problem 2.2 Inventing a passing candidate

Can you invent a second candidate for the area that has correct dimensions and passes the $a = 0$, $b = 0$, and $a = b$ tests?

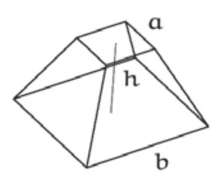
Problem 2.3 Generalization

Guess the volume of an ellipsoid with principal radii a, b, and c.

2.3 Solid geometry: The volume of a truncated pyramid

The Gaussian-integral example (Section 2.1) and the ellipse-area example (Section 2.2) showed easy cases as a method of analysis: for checking whether formulas are correct. The next level of sophistication is to use easy cases as a method of synthesis: for constructing formulas.

As an example, take a pyramid with a square base and slice a piece from its top using a knife parallel to the base. This truncated pyramid (called the frustum) has a square base and square top parallel to the base. Let h be its vertical height, b be the side length of its base, and a be the side length of its top.



What is the volume of the truncated pyramid?

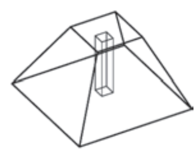
Let's synthesize the formula for the volume. It is a function of the three lengths h, a, and b. These lengths split into two kinds: height and base lengths. For example, flipping the solid on its head interchanges the meanings of a and b but preserves h; and no simple operation interchanges height with a or b. Thus the volume probably has two factors, each containing a length or lengths of only one kind:

$$V(h, a, b) = f(h) \times g(a, b). \tag{2.4}$$

Proportional reasoning will determine f; a bit of dimensional reasoning and a lot of easy-cases reasoning will determine g.

What is f: How should the volume depend on the height?

To find f, use a proportional-reasoning thought experiment. Chop the solid into vertical slivers, each like an oil-drilling core; then imagine doubling h. This change doubles the volume of each sliver and therefore doubles the whole volume V. Thus $f \sim h$ and $V \propto h$:



$$V = h \times g(a, b). \tag{2.5}$$

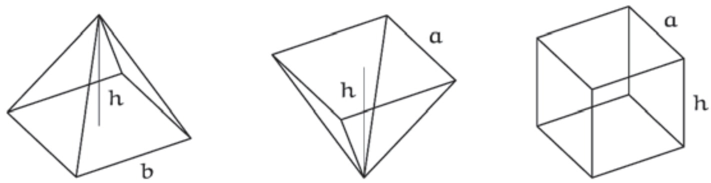
What is g: How should the volume depend on a and b?

Because V has dimensions of L^3, the function $g(a, b)$ has dimensions of L^2. That constraint is all that dimensional analysis can say. Further constraints are needed to synthesize g, and these constraints are provided by the method of easy cases.

2.3.1 Easy cases

What are the easy cases of a and b?

The easiest case is the extreme case $a = 0$ (an ordinary pyramid). The symmetry between a and b suggests two further easy cases, namely $a = b$ and the extreme case $b = 0$. The easy cases are then threefold:



$a = 0$ $b = 0$ $a = b$

When $a = 0$, the solid is an ordinary pyramid, and g is a function only of the base side length b. Because g has dimensions of L^2, the only possibility for g is $g \sim b^2$; in addition, $V \propto h$; so, $V \sim hb^2$. When $b = 0$, the solid is an upside-down version of the $b = 0$ pyramid and therefore has volume $V \sim ha^2$. When $a = b$, the solid is a rectangular prism having volume $V = ha^2$ (or hb^2).

A
Sample Appendix

Here is appendix text.

A.1 Sample appendix section

Here is appendix text.

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